Extra Practice Problems

Here are some extra practice problems on topics that were popular on the Google Moderator site. We'll release solutions to these problems on Monday.

Problem One: Binary Relations

Suppose that R and S are binary relations over some set A. The **composition of R and S**, denoted $S \circ R$, is the relation defined as follows:

For any $x, y \in A$, $x(S \circ R)y$ iff there is some $z \in A$ such that xRz and zSy.

One interesting special case to consider is the composition of a relation with itself. Given a binary relation R, the relation $R \circ R$ is defined as follows: $x(R \circ R)y$ iff there is some z such that xRz and zRy. We use the notation R^2 to denote $R \circ R$.

Prove that $R = R^2$ if R is reflexive and transitive. Recall that $R = R^2$ iff for any $x, y \in A$, we have that xRy iff xR^2y.

Problem Two: Diagonalization

As you'll see on Friday's lecture, a co-recognizer for a language L is a TM M such that for any string w, the TM M rejects w iff $w \notin L$. By definition, a language L is in co-**RE** iff there is a co-recognizer M for L.

Let $L_{\text{co-D}} = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ rejects } \langle M \rangle \}$. Prove that $L_{\text{co-D}} \notin$ co-**RE**. *(Hint: use a proof by diagonalization.)*

Problem Three: First-Order Logic

In what follows, let's assume the domain of discourse is a nonempty set of people, so all quantifiers range over people.

Consider the predicate $Drinks(p)$, which says that p is currently drinking. The *drinker's paradox* is the following statement:

$$\exists p.\ (Drinks(p) \to \forall q.\ Drinks(q))$$

This says "there is someone where if that person is drinking, then *everyone* is drinking."

i. Explain why the above statement is always true, regardless of who's drinking.

ii. Is the above statement equivalent to the following statement?

$$(\exists p.\ Drinks(p)) \to (\forall q.\ Drinks(q))$$

Problem Four: Regular Expressions

Below are some alphabets and languages over those alphabets. Design a regular expression for each of those languages.

i. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ w \mid w \neq \varepsilon$ and w starts and ends with the same symbol. $\}$ Write a regular expression for L.

ii. Let $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ and let $L = \{ \langle w_1, w_2 \rangle \mid w_1, w_2 \in \Sigma^* \}$. That is, L is the set of all pairs of strings encoded using the encoding format used on Problem Set Seven. Write a regular expression for L.

iii. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w$ has an even number of **b**'s $\}$. Write a regular expression for L.

Problem Five: Context-Free Grammars

Below are some alphabets and languages over those alphabets. For each of the languages, write a context-free grammar for that language.

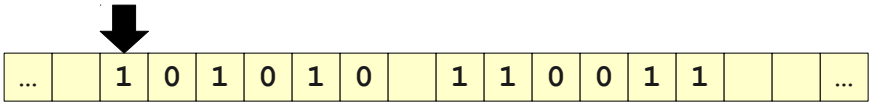
i. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ \mathtt{a}^n\mathtt{b}^m \mid n, m \in \mathbb{N}$ and $n \neq m \}$. Write a CFG for L.

ii. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ w \in \Sigma^* \mid$ there are more **a**'s than **b**'s in $L \}$. Write a CFG for L.

iii. (Challenge problem!) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{ xy \mid x, y \in \Sigma^*, |x| = |y|$, and $x \neq y \}$. In other words, L is the language of all strings that whose first half is not the same as its second half. Write a CFG for L.

Problem Six: Turing Machines

Let $\Sigma = \{0, 1\}$. Design a Turing machine that, given as input two equal-length strings separated by a blank character and surrounded by infinitely many blanks, computes the XOR of those two strings and ends with that XORed string written on the tape, surrounded by infinitely many blanks. For example, given this initial configuration:



your Turing machine would end in this configuration: